Two-Dimensional Interception Problem

Karl Voskuil

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1 Problem

The monster M wants to catch the elusive tofubeast T.

M and T are two points on a plane. T is moving in direction a_T (an angle clockwise relative to 0° north) at a constant velocity v_T. a_S is the direction from M to T (angle of sight, relative to the same 0° north), d is the distance between them, and v_M is the constant velocity of point M. All these are given, and shown in Figure 1. Angles θ and α are derived (from givens, and from the unknown direction a_M in which M will travel):

$$\theta = a_T - (a_S - 180)$$
$$\alpha = a_S - a_M$$

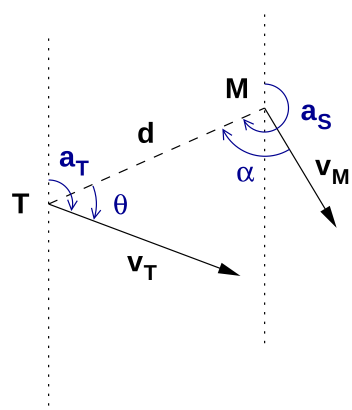


Figure 1: The physical problem.

Suppose a point I such that if T and M continue at their constant speeds and directions they will intercept at time t_0. In other words, at any point in time

t, with $\Delta t = t_0 - t$ seconds until interception, the points can be represented as shown in Figure 2, where r_{XY} is the rate at which points X and Y are approaching. (Notice in particular the way assumptions of the problem are captured: an extant I, and θ and α constant over time.)

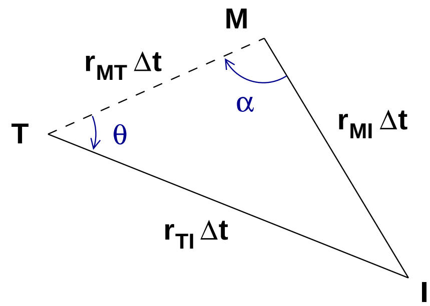


Figure 2: A mathematical representation of the problem.

The problem:

1. Make an expression for α in terms of v_T, v_M, and θ.
2. How many points I are there for a given v_T, v_M, and θ?

2 Solution

2.1 Formula

Drop a height h from I down to side MT. Then

$$\sin\theta = \frac{h}{r_{TI}\Delta t}$$
$$\sin\alpha = \frac{h}{r_{MI}\Delta t}$$

and so

$$\sin\theta \cdot r_{TI}\Delta t = \sin\alpha \cdot r_{MI}\Delta t \qquad \text{(Law of Sines)}$$

Through time the angles of the representation triangle are constant and the sides are proportional by some factor $\Delta t_2/\Delta t_1$: when solving for the angles, time may be eliminated from both the equation and the representation of the problem.

$$\sin\theta \cdot r_{TI} = \sin\alpha \cdot r_{MI}$$

Since point I is not moving and thus $v_T = r_{TI}$ and $v_M = r_{MI}$, α can be expressed by the equation

$$\sin\alpha = \sin\theta \cdot \frac{v_T}{v_M}$$

Mathematically, the solution for α is ambiguous, since $\sin\alpha = \sin(180 - \alpha)$. However, in the physical situation it is clear that the α desired is always acute; otherwise M would try and catch T by running away from it rather than towards it—a sometimes successful but always inefficient strategy. (The two solutions for α are pictured in Figure 3; α_1 is always preferable to α_2, because a smaller α means M is heading more directly towards the current location of T. Alternately: according the representation, a smaller α maximizes the length of MT, which represents r_{MT}, the closure rate between points M and T.) Assuming

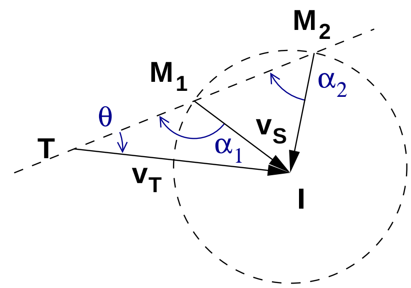


Figure 3: Ambiguous interception (mathematical representation).

the implementation of the arcsin function returns only the acute or right angle, the best solution for α can be simply stated

$$\alpha = \arcsin\left(\sin\theta \cdot \frac{v_T}{v_M}\right) \tag{1}$$

2.2 Formula Values

Once the ambiguous case for α has been eliminated by the definition of the arcsin function, there might be zero or one values α (and corresponding points I) for a given v_T, v_M, and θ.

(If the distance between M and T is zero, then a single point of interception exists at the shared point—or an infinite number of points of interception over time if in addition $v_M = v_T$. These special cases are excluded from all the considerations below.)

Equation (1) is undefined when v_M is zero. If T happens to be headed directly for M with a non-zero velocity (that is, $a_T = a_S - 180$, which means θ is zero, and v_T is non-zero), then a point of interception I will exist at point M. Otherwise, no point of interception will exist. In either case, the value of α is irrelevant to the physical problem since point M does not move.

Equation (1) is undefined when the parameter of the arcsin function is less than -1 or greater than 1. Consider the parameter of the arcsin function: $\sin\theta$ is a number between -1 and 1 which will modify the speed ratio to determine how far M has to turn from its line of sight in order to try to intercept T: a faster T and a θ close to $90°$ lead to a more radical turn. If, for instance, T is headed directly toward or directly away from M, then $\theta = 0°$ or $180°$, $\sin\theta = 0$, and the relative speeds of M and T are irrelevant to α: the intercept angle α will be 0 (i.e. M should run directly towards T to try to intercept). If, on the other hand, T is running directly broadside to M's line of sight, $\theta = 90°$, then M is going to have to turn quite a ways from its line of sight to try and intercept T: how far, exactly, is determined only by the ratio of their speeds, since $\sin\theta$ will be 1.

So, when the ratio of the velocities modified by the relative angle of their vectors is less than -1 or greater than 1, T will escape from M off to one side or the other, and no point I is defined. This doesn't describe all possible escapes of T; rather, it describes the escapes where T is running obliquely—at something like a right angle—across M's line of sight. Such an escape has an understandable representation when θ is less than $90°$, as shown by θ_1 in Figure 4: T is moving so fast (relative to M) that even though T is headed in the general direction of M, M still can't close the gap, even with a maximum α of $90°$. (Because of the ambiguity expressed by $\sin\theta = \sin(180 - \theta)$, a similar escape where θ is greater than $90°$ is not easily pictured—see $\theta_2 = (180 - \theta_1)$ in the figure. The mathematical representation begins to have little correspondence with the physical sitatution, and it seems strange that M_2 would flip around to the other side of T. This difficulty of representation is discussed more fully in the next paragraphs.)

A description of all possible escapes cannot use equation (1) as its sole criterion, since in many cases the equation will happily yield an α even though an interception is impossible (and even though the reprentation expressed by the formula has long since stopped having any correspondence with the physical situation). The reflection case ($\theta_2 > 90°$) of the previous paragaph begins to illustrate the failure of the representation (that is, the failure of the formulaic solution for α) when v_M is not large enough for an interception.

Figure 5 further clarifies the potential difficulty, and combines a physical representation (at a certain point in time) in black, and the corresponding mathematical representation (for the α equation) in red. If the black M and T are points in a plane, then θ is the angle greater than $90°$ between M's line of sight to T and T's velocity vector v_T. The formula treats θ the same as $180 - \theta$, and since v_M is so small, the red triangle on which the formula is based must be drawn with the acute $180 - \theta$. An acute α is then chosen by the arcsin function in the formula (for a height dropped from I to MT), rather than the formulaically equivalent $180 - \alpha$ pictured as angle TMI). If this α were translated back into the physical situation (as shown by the gray angle and vector at point M), M would be heading in an apparently arbitrary direction. Magic formula (1)'s result is not to be trusted.

Consider, on the other hand, a representation of the problem when θ is greater than $90°$ but v_M is at least as large as v_T, as shown in Figure 6. Now when

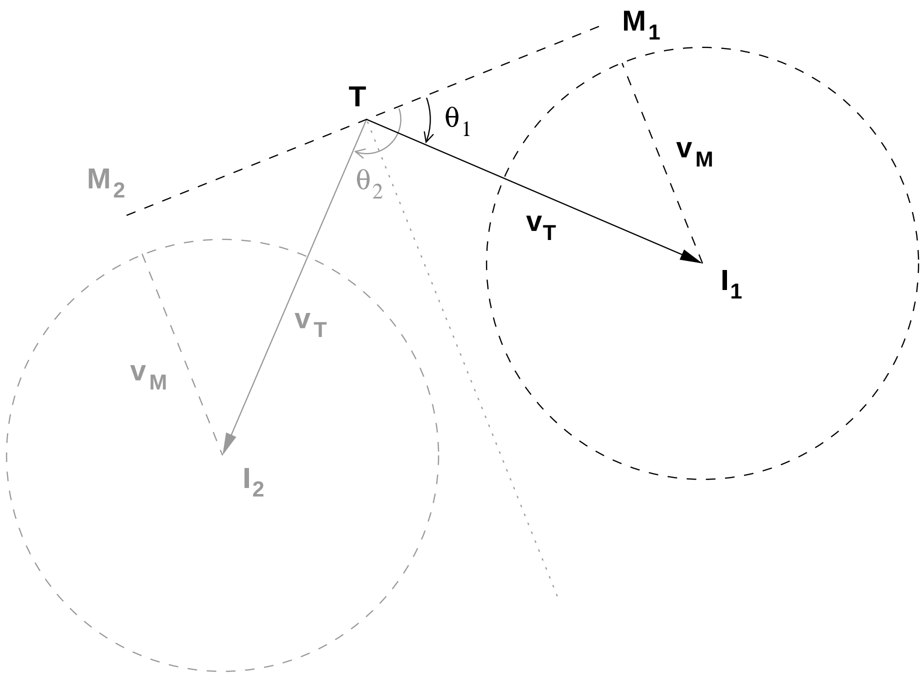


Figure 4: An oblique escape; α undefined. The black lines show an example acute θ; the gray lines the reflected obtuse θ.

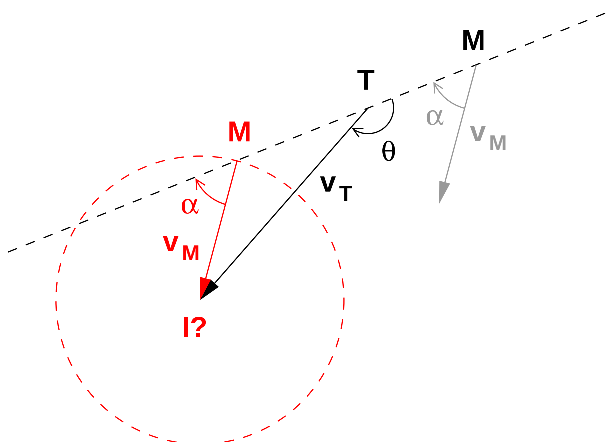


Figure 5: A strange pursuit by M; α defined but useless. The red lines show the mathematical representation; the gray lines show the calculated α in the physical representation.

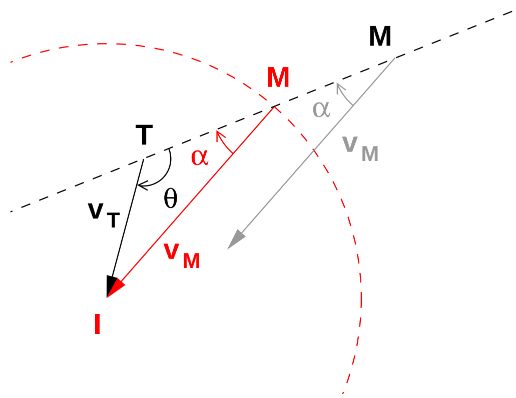


Figure 6: A sensible α with an obtuse θ. The red lines show the mathematical representation; the gray lines show the calculated α in the physical representation.

the red mathematical solution is translated back into the physical situation, the calculated α makes sense—and it appears M will intercept T sometime in a little less than two ticks. Formula (1) yields a sensible α for $\theta > 90°$ (or $\theta < -90°$) all the way up to $v_M = v_T$. (When $v_M = v_T$, then $\alpha = 180° - \theta$, and T is instructed to run parallel to T. Sure, M won't make the interception by running parallel—but at least it's a sensible direction in which to run.)

The two illustrations in Figure 7, then, are the physical representations of the general case for non-interception. When $\theta < 90°$, then formula (1) will be undefined. When $\theta >= 90°$, either formula (1) will be undefined, or v_T will be greater than v_M, or both. In either situation, an interception is impossible according to the assumptions of the problem; but otherwise, formula (1) will give a correct interception value.

2.3 Sensible Solutions for the Escape Cases

Even when an interception is impossible, the monster might still try to chase the tofubeast. What, then, is the best α? The ghosted α of the escape illustrations in Figure 7 might appear to be the only natural choice, but it's a little misleading: the whole point is that a constant α cannot be found, and so the illustrations are only geometrically valid for a certain point in time. The choice of α will depend on factors outside the scope of the problem. Some ideas:

- If M expects T to run straight all the way to a distant finish line, the best α might be defined as the angle that gets M to the finish line with the least amount of lag behind M—which would be $\alpha = 180 - \theta$, or a parallel course.

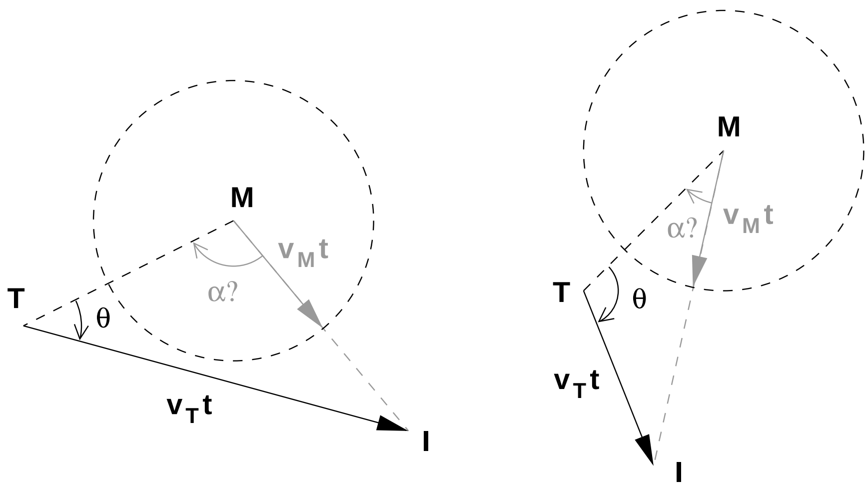


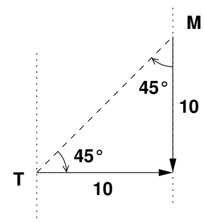
Figure 7: Two cases representing all possible escapes. On the left, $\theta < 90°$; on the right, $\theta >= 90°$.

- If M expects T to reverse direction sometime soon, M might do best to head directly towards the current location of T (at any given point in time): $\alpha = 0$.
- A good general-purpose α would be the angle at which M should travel to intercept T if M's velocity were just large enough to effect an interception at some point in the future (or at least maintain a distance if behind). For $\theta < 90°$, $\alpha = 90°$; for $\theta \geq 90°$, $\alpha = 180 - \theta$. As with any α chosen between 0 and $180 - \theta$ for a non-interception, the problem will change with each tick: θ will increase, and point M will curve around behind T in pursuit.
- The illustrations above are only valid for a certain point in time during the chase: the geometry of θ and the theoretical point I will change, making a straight shot to I impossible (and therefore α undefined). But the ghosted α and straight $v_M t$ seem to imply a possible solution over time: take a straight path for a little while by (arbitrarily) picking a time in the future, calculating the distance T will travel assuming constant velocity and direction, and then heading toward that point (regardless of v_M). Of course, once that future time is past, a new α will be needed, and the path of M over time, if not curved, will then be segmented. (Notice this solution and the second are actually quite similar: this α will probably end up being only a few degrees off 0.)

3 Examples

1. *Acute approach.* The Tofubeast is running due east at 10 mph. Monster is northeast of the Tofubeast and can run at 10 mph.

$$a_T = 90° \text{ [east]}$$
$$a_S = 225° \text{ [southwest]}$$
$$v_T = v_M = 10 \text{ mph}$$

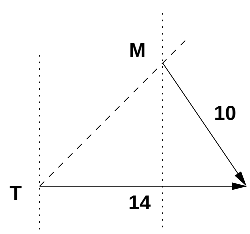


Solution

$$\theta = a_T - (a_S - 180) = 45°$$
$$\alpha = \arcsin\left(\sin\theta \cdot \frac{v_T}{v_M}\right) = 45°$$
$$a_M = a_S - \alpha = 180° \text{[south]}$$

Variations

- if $v_T = 5$ then $a_M = 204.3°$ [south-southwest]
- if $v_T = 14$ then $a_M = 143.1°$ [southeast]

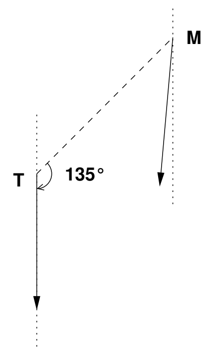


- if $v_T = 15$ then $a_M =$ *undefined*
  (notice if $\alpha = 90°$, then $10 \cdot \sqrt{2} = 14.14 < 15$)

2. *Obtuse approach.* The Tofubeast is running due south at 10 mph. Monster is northeast of the Tofubeast and can run at 11 mph.

Solution

$$\theta = 135^\circ$$
$$\alpha = 40^\circ$$
$$a_M = 185.0^\circ$$



Variations

- if $v_M = 15$ then $a_M = 196.9^\circ$
- if $v_M = 20$ then $a_M = 204.3^\circ$
- if $v_M = 30$ then $a_M = 211.4^\circ$
- if $v_M = 9$ then $a_M =$ *undefined*

Notice in the last variation, where $v_M = 9$, then according to equation (1):

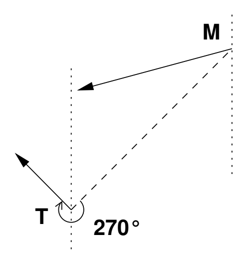
$$\alpha = \arcsin\left(\sin\theta \cdot \frac{v_T}{v_M}\right) = \arcsin\left(0.707 \cdot \frac{10}{9}\right) = 51.8^\circ$$
$$a_M = 225^\circ - \alpha = 173.2^\circ?$$

This would be a poor choice of α: M is instructed to head away from T in order to intercept, which makes no sense. But $\theta > 90^\circ$ and $v_T > v_M$, which is the special case of non-interception even when equation (1) is defined, and so a_M is undefined.

3. *Negative-side approach.* The Tofubeast is running northwest at 5 mph. Monster is northeast of the Tofubeast and can run at 10 mph.

Solution

$$\theta = 270^\circ \text{ or } -90^\circ$$
$$\alpha = -30^\circ$$
$$a_M = 255.0^\circ$$



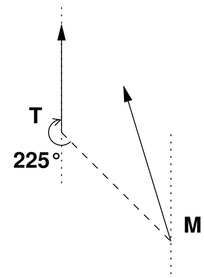
Variation

- if $a_T = 0^\circ$, $v_T = 10$ then $\theta = -45^\circ$, $a_M = 270^\circ$

4. *Angle of sight.* The Tofubeast is running north at 10 mph. Monster is southeast of the Tofubeast and can run at 15 mph.

Solution

$$\theta = 225^\circ \text{ or } -135^\circ$$
$$\alpha = -28.1^\circ$$
$$a_M = 343.1^\circ$$



Variations

- if $a_S = 345^\circ$ then $\theta = -165^\circ$, $a_M = 354.9^\circ$
- if $a_S = 0^\circ$ then $\theta = 180^\circ$, $a_M = 0^\circ$
- if $a_S = 45^\circ$ then $\theta = 135^\circ$, $a_M = 16.9^\circ$
- if $a_S = 90^\circ$ then $\theta = 90^\circ$, $a_M = 48.2^\circ$
- if $a_S = 115^\circ$ then $\theta = 65^\circ$, $a_M = 77.8^\circ$